# Calculus 2

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Calculus 2

# Exercices 4

# Infinite Series

## Evaluate

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

## Solution

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots$$

$$0.3+0.03+0.003+0.0003+\cdots = 0.333\ldots$$

$$=\frac{1}{3}$$

$$\frac{\overbrace{\dfrac{3}{10}}^{a}}{1-\underbrace{\dfrac{1}{10}}_{r}}=\frac{\dfrac{3}{10}}{\dfrac{9}{10}}=\frac{3}{9}=\frac{1}{3}$$

## Solution

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

$$\frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

($1 \leftarrow a$, $-\frac{1}{2} \leftarrow r$)

## Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+ \cdots$$

$$S_3 = 1 - \frac{1}{4} \qquad S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = 1$$

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Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

## Solution

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

**Solution** For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots$$
$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

**EXAMPLE** Express the repeating decimal 5.232323 . . . as the ratio of two integers.

**Solution** From the definition of a decimal number, we get a geometric series

$$5.232323\ldots = 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{1/(1-0.01)} \qquad \begin{array}{l} a = 1, \\ r = 1/100 \end{array}$$

$$= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99}$$

Find the sums of the following series.

(a) $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

(b) $\sum_{n=0}^{\infty} \frac{4}{2^{n}}$

# Solution

**(a)** $$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$ Difference Rule

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)}$$ Geometric series with $a = 1$ and $r = 1/2, 1/6$

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

**(b)** $$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$$ Constant Multiple Rule

$$= 4\left(\frac{1}{1-(1/2)}\right)$$ Geometric series with $a = 1, r = 1/2$

$$= 8$$

# Using the *n*th-Term Test for Divergence

$$\sum_{n=0}^{\infty} 2^n;$$

$$\sum_{n=1}^{\infty} \frac{n!}{2n!+1},$$

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## Solution

**a.** For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the $n$th term is not 0, and the series diverges.

**b.** For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}.$$

So, the limit of the $n$th term is not 0, and the series diverges.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}.$$

**Solution** Disregarding all but the highest powers of $n$ in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$ Convergent $p$-series

Because

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2+1}\right)\left(\frac{n^{3/2}}{1}\right)\\ &= \lim_{n\to\infty} \frac{n^2}{n^2+1}\\ &= 1\end{aligned}$$

you can conclude by the Limit Comparison Test that the series converges.

# Find the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

## Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$1 = (A+B)n + 2A$$

$$A = \frac{1}{2}, B = -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$S_n = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+2}\right)\right]$$

$$S_n = \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right) \to \frac{1}{2}\left(1+\frac{1}{2}\right)$$

$$S = \frac{3}{4}$$

The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$

$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdot \cdot \cdot$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$.

Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1 - r} = \frac{3}{1 - (1/2)} = 6.$$

The geometric series

$$\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$.

Because $|r| \geq 1$, the series diverges.

# $0.999... = 1?$

## Solution

$$0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}\ldots$$

$$= \frac{9}{10} + \frac{9}{10}\left(\frac{1}{10}\right) + \frac{9}{10}\left(\frac{1}{100}\right) + \frac{9}{10}\left(\frac{1}{1000}\right) + \ldots$$

$$= \frac{9}{10} + \frac{9}{10}\left(\frac{1}{10}\right)^1 + \frac{9}{10}\left(\frac{1}{10}\right)^2 + \frac{9}{10}\left(\frac{1}{10}\right)^3 + \ldots$$

$$0.999\ldots = \left(\frac{9}{10}\right)\left(\frac{1}{1-\frac{1}{10}}\right)$$

$$= \left(\frac{9}{10}\right)\left(\frac{1}{\left(\frac{9}{10}\right)}\right)$$

$$= \left(\frac{9}{10}\right)\left(\frac{10}{9}\right) = 1$$

$$1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots=\ ?$$

$$1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots=\ ?$$

# Solution

$$S=1+\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots=1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots$$

$$1+(\tfrac{1}{1}-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{4}-\tfrac{1}{5})+\ldots$$

$$1+1=\boxed{2}$$

# Exercises

If $\{S_n\}$ is the sequence of partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$, then $\lim_{n\to\infty} S_n$

(a) is equal to $\frac{3}{2}$

(b) is equal to 1

(c) is equal to $\frac{1}{2}$

(d) is equal to 0

(e) does not exist

If $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n}}-\frac{1}{\sqrt[3]{n+1}}\right)$, then $S_n =$

(a) $1-\frac{1}{\sqrt[3]{n}}$

(b) $\frac{1}{\sqrt[3]{n}}-\frac{1}{\sqrt[3]{n+1}}$

(c) $\frac{n}{\sqrt[3]{n}}$

(d) $\frac{1}{\sqrt[3]{2}}-\frac{1}{\sqrt[3]{n+1}}$

(e) $1-\frac{1}{\sqrt[3]{n+1}}$

The series $\sum_{n=2}^{\infty} 3^{n+1} \cdot 2^{1-2n}$ is

(a) divergent

(b) convergent and its sum is $27/2$

(c) convergent and its sum is $9/4$

(d) convergent and its sum is $3/2$

(e) convergent and its sum is $27/8$

The series $\sum_{n=0}^{\infty} \frac{e^{1-2n}}{(\sqrt{2})^{2-2n}}$ is

(a) Convergent and its sum is $\frac{e^3}{2e^2-4}$

(b) Convergent and its sum is $\frac{1}{e^3}$

(c) Convergent and its sum is $\frac{e}{2}$

(d) Convergent and its sum is $\frac{e^2}{e^2-2}$

(e) Divergent

Find the value of $b$ for which

$$\sum_{n=0}^{\infty} e^{nb} = 1 + e^b + e^{2b} + e^{2b} + \cdots = 9$$

(a) $\ln\left(\frac{8}{9}\right)$

(b) $\frac{-e}{9}$

(c) $\ln\left(\frac{1}{9}\right)$

(d) $\ln\left(\frac{9}{8}\right)$

(e) $e\ln\left(\frac{1}{9}\right)$

The sum of the series $\sum_{n=1}^{\infty}[\frac{3}{n(n+1)}+\frac{1}{2^n}]$ is equal to

(a) 4

(b) 3

(c) 2

(d) 1

(e) 5

The series $\sum_{n=1}^{+\infty} \frac{2^n + (-1)^{n-1}}{3^n}$

(a) diverges

(b) converges and its sum is $\frac{3}{4}$

(c) converges and its sum is 2

(d) converges and its sum is $\frac{2}{3}$

(e) converges and its sum is $\frac{9}{4}$

$$\sum_{n=1}^{\infty} \frac{1+(-2)^n}{3^n} =$$

(a) 0.001

(b) 0.01

(c) 0.1

(d) 1.1

(e) 1.01

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**Thank you for your attention**